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PERIODIC SOLUTIONS OF SYSTEMS WITH GYROSCOPIC FORCES*

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The lower limit for the number of periodic solutions of the equations of motion of a material point in n-dimensional Euclidean space under the effect of potential and gyroscopic forces is proved.

We consider a system with the gyroscopic forces /1/

$$(A(t)x^\cdot)^\cdot = \Gamma x + U_x(x, t), \quad x \in R^n \tag{1}$$

where $A(t)$ is a symmetric positive-definite matrix, 2π-periodically continuously dependent on time, Γ is a constant skew-symmetric matrix of the gyroscopic forces, and the potential U depends 2π-periodically continuously on time, has continuous second derivatives with respect to the space variables and is periodic in them, for example

$$U(x+k, t) \equiv U(x, t) \tag{2}$$

for all integer vectors $k \in Z \subset R^n$.

Theorem. If the system

$$(A(t)x^\cdot)^\cdot = \Gamma x^\cdot \tag{3}$$

has no non-constant 2π-periodic solutions, then system (1) has no less than $n+1$ different 2π-periodic solutions, and when multiplicity is taken into account, no less than 2^n. Solutions differing by a shift in the period of the potential are considered to be identical.

The conditions of the theorem mean that $A(t)x^\cdot = \Gamma x$ has no Floquet multipliers equal to one. If the potential U is small, then the assertion of the theorem can be obtained by methods of Poincaré perturbation theory.

System (1) is Lagrangian with the Lagrange function

$$L(x, x^\cdot, t) = {}^1/_2 (A(t)x^\cdot, x^\cdot) + {}^1/_2 (\Gamma x^\cdot, x) + U(x, t) \tag{4}$$

We will seek 2π-periodic solutions of system (1) as critical points of the Hamilton action functional

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$$F(x) = \int_0^{2\pi} L(x(t),\ x^{\cdot}(t),\ t)\,dt \tag{5}$$

in the set of 2π-periodic curves $t \mapsto x(t) \in R^n$. The domain of definition of the functional (5) will be refined later.

If there are no gyroscopic forces, function (4) is periodic in the space variables, function (5) is defined in a set of curves on the n-dimensional torus $T^n = R^n/Z^n$ and the assertion of the theorem results from the results of the calculus of variations (Morse's theory). In the general case, functional (5) has no lower limit so that the ordinary Morse theory is not applicable. An analogue to Morse's theory /2/ has been developed for unbounded and multivalued functionals, however, the results of /2/ are inapplicable in this case since the constant curves are not local minimum points of the functional (5). The proof of the theorem is based on the ideas in /3/.

Let H be a Hilbert space of 2π-periodic functions $t \mapsto x(t) \in R^n$ with components of the class L^2 and the scalar product

$$\langle x, y\rangle = \int_0^{2\pi} (x(t), y(t))\,dt$$

Let $X \subset H$ be the domain of definition of the linear selfadjoint operator A corresponding to system (3)

$$(Ax)(t) = -(A(t)\,x^{\cdot}(t))^{\cdot} + \Gamma x^{\cdot}(t) \tag{6}$$

i.e., the set $x \in H$ is such that $Ax \in H$. The set X has the structure of a Hilbert space, copact in H, the imbedding $X \subset H$ is completely continuous, and (5) defines a functional of the class C^2

$$F(x) = {}^1/_2\langle Ax, x\rangle + \bar{U}(x) \tag{7}$$

in X, where

$$\bar{U}(x) = \int_0^{2\pi} U(x(t), t)\,dt \tag{8}$$

is a functional of class C^2 in H. The critical points of functional (7) are in one-to-one correspondence with the 2π-periodic solutions of the system (1).

We represent the argument $x \in X$ of the functional (7) in the form $x = \bar{x} + \xi$, where $\bar{x} \in R^n$ is the mean value of x and ξ is an element of the set $X_0 \subset X$ of functions with zero mean. In the new $\bar{x}, \xi$ variables

$$F(x) = F(\bar{x}, \xi) = {}^1/_2\langle A\xi, \xi\rangle + \bar{U}(\bar{x} + \xi) \tag{9}$$

By virtue of (2), (8) and (9), we have

$$F(\bar{x} + k, \xi) \equiv F(\bar{x}, \xi) \tag{10}$$

for all $k \in Z^n$. Consequently, (9) defines a functional of class C^2 on $T^n \times X_0$, where $T^n = R^n/Z^n$ is an n-dimensional torus. By the condition of the theorem, the kernel of the operator (6) consists of the set of constant functions $R^n \subset X$ such that the quadratic form in (9) is non-degenerate.

We reduce the search for critical points of the functional (9) to an investigation of functions of a finite number of variables by the Lyapunov-Shmidt method. We select

$$a > \max_{x, t} \| U_{xx}(x, t)\| \tag{11}$$

According to the Sturm-Liouville theorem /4/, the selfadjoint operator A has a compact resolvent and its purely real point spectrum has no accumulation points except $+\infty$. Let y be the orthogonal projection of the vector $x \in H$ on the subspace $Y \subset H$ corresponding to the part of the spectrum of the operator A lying in $[a, +\infty)$ and z the projection on the additional subspace $Z \subset H$. The subspaces Y and Z are invariant with respect to A, where Z is finite-dimensional. The operator $A\,|\,Y$ has the compact inverse $A^{-1}\colon Y \to X \in H$, where $\|A^{-1}y\| \leqslant a^{-1}\|y\|$ for all $y \in Y$. In the new y, z variables formula (7) becomes

$$F(x) = F(y + z) = {}^1/_2\langle Ay, y\rangle + {}^1/_2\langle Az, z\rangle + \bar{U}(y + z) \tag{12}$$

The critical points of the functional F are determined from the equations

$$\nabla_y F(y + z) = 0, \quad \nabla_z F(y + z) = 0 \tag{13}$$

The first of Eqs.(13) is equivalent to the equation

$$y + A^{-1}\ \nabla_y \bar{U}(y + z) = 0 \tag{14}$$

By virtue of (11), it follows from the theorem on implicit functions or the principle of compressed mappings that (14) has the unique solution $y = h(z)$, where $h\colon Z \to X$ is a function of class C^1. We set $f(z) = F(h(z) + z)$. By construction, f is a function of class C^2 in the finite-dimensional space Z and its critical points are in one-to-one correspondence with the critical points of the functional F.

By virtue of (10) the function f defines a function of the class C^2 on $T^n \times Z_0$, where Z_0 is a set of functions from Z with zero mean. According to (12), setting $z = \bar{z} + \zeta$, we have

$$f(\bar{z}, \zeta) = {}^1/_2\langle A\zeta, \zeta\rangle + g(\bar{z}, \zeta); \quad \bar{z} \in T^n, \quad \zeta \in Z_0$$

where $\langle A\zeta, \zeta\rangle$ is a non-degenerate quadratic form on $Z_0 = R^N$, and the partial derivatives of the function g are bounded for $\|\zeta\| \to \infty$. From this and the results in /3/ the assertion of the theorem follows.

The theorem can be extended to the case when the potential U is invariant relative to any crystallographic group G of transformations of the space R^n. In this case, Z^n must be replaced by G in the proof, and the Lyusternik-Shnirel'man category of the space R^n/G is the lower bound of the number of periodic solutions.

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ON TRANSONIC EXPANSIONS*

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The problem of finding the particular solutions of the linearized inhomogeneous transonic equations appearing in the transonic expansions, expressed explicitly in terms of the fundamental solution of the Karman-Fal'kovich (KF) equation, is discussed.

When the procedure of transonic expansion is used, e.g. in the thin-body theory /1/, the solutions of the equations of gas dynamics have the form of series in powers of a small parameter characterizing the measure of the deviation of the flow in question from homogeneous sonic, or a nearly sonic flow. To a first approximation, the non-linear KF equation has to be solved /2, 3/, and inhomogeneous linearized KF equations whose right-hand sides depend on the preceding terms are obtained for the higher-order approximations. It is convenient to have available an explicit expression for the particular solutions written in terms of the fundamental solution. Thus in /4/ two examples are given of determining the first correction in the theory of small perturbations for the axisymmetric flows of a compressible fluid when the correction is expressed in terms of the fundamental solution without taking into account its specific structure, and the uniqueness of such results is noted. The first-order correction to the solution of the KF equation was obtained in /5/.

In the case of plane parallel flow the KF equation reduces, in the hodograph plane, to the linear Tricomi equation, and the procedure of transonic expansion enables one, as was shown in /6, 7/, to determine particular solutions for any order of approximation. From this it follows that when transonic expansions are used, particular solutions of a general type can be obtained in the physical plane for the i-th approximation. The present communication does not demonstrate the procedure of passing from the hodograph expansions to expansions in the physical plane, but gives the following straightforward result: the first correction which is the same as that obtained in /5/, and the second correction. The fact that curvilinear integrals appear in the second correction but not in the first, is of interest.

In the case of an axisymmetric flow the first correction to the solution of the KF equation has the same form as in the plane parallel case. However, attempts, using the analogy with the plane-parallel case, to find the second correction in general form, have proved

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